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CALCULATION OF THE BENDING STRESSES IN HELICOPTER ROTOR BLADES

By P. de Guillenchmidt

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CALCULATION OF THE BENDING STRESSES IN HELICOPTER ROTOR BLADES*

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INTRODUCTION

The problem of determining the static and dynamic stresses on the blades of rotary-wing aircraft in forward flight has occupied the attention of many engineers of every country for a long time. Up to within the last few years, however, no satisfactory solution has been found for blades having a distribution of mass and rigidity varying along the span. In France, Mr. Dorand had for many years (1932 with the Breguet-Dorand gyroplane) used a graphical method which gave satisfactory results; however, it was too long and called for precise diagrams on account of the graphic double derivations involved. More recently, American engineers have developed several methods which, while affording correct solutions in the general case, also require a volume of calculation, which increases at an appalling rate when the number of points examined on the blade are to be increased or higher harmonics for the external forces acting on the blade are to be introduced. Accordingly, it has been necessary to introduce more approximate methods, and it is these methods which are usually employed in design.

The purpose of the present report is to describe a comparatively rapid method of calculation which gives a correct theoretical solution of the problem in the most general case. This method is the result of collaboration between the Bureau of Calculation of the Helicopter Division of the National Societies of Airplane Construction for South Eastern and for Central France, set up within the Committee of Rotating Wing Units of the French Association of Aeronautical Engineers and Technicians, (A.F.I.T.A.), on the instigation of Col. Garry, Chief of the Section of Rotating Wing Units of the Technical Service Division of the Air Ministry.

The method is based on the analysis of the properties of a vibrating beam, and its uniqueness lies in the simple solution of the differential equation which governs the motions of the bent blade. It is applicable, whatever the limiting conditions may be (blades hinged, blades clamped, blades fitted with dampers, etc. . . .). It requires, strictly speaking, the preliminary calculation of the natural frequencies and mode shapes of

*"Calcul en Flexion de Pales de Giravions." S.N.C.A.C. Report, Document He3-0.03, December 23, 1948.

the blade in rotation. This calculation can be reduced, however, as will be shown later, in a certain number of cases, to the calculation of the natural frequency of the first mode[1], which reduces the calculation required to some extent.

For the explanation of the method, let us take the case of a hinged blade in flapping motion and impose the usual restrictive asumptions, which are:

(a) The twisting deflections of the blade are negligible.

(b) The blade is rigid in its plane, that is to say, the drag deflections are negligible with respect to the deflections normal to the plane of the blade.

(c) The deflections and the flapping angles are small, hence their higher powers can be disregarded and we may assume

$$\cos \beta \sim 1 \qquad \sin \beta \sim \tan \beta \sim \beta$$

(d) The bending deflections of the blade do not influence the aerodynamic forces acting on the blade. Included, however, is a term for the aerodynamic damping due to the fact that the vibration of the blade produces, for each element, a change in the relative velocity, and consequently of the angle of attack.[2]

SYMBOLS

R	radius of rotor
r	distance of blade element from axis of rotation
x	abscissa, along axis OX of the rigid blade, of an element of the elastically deflected blade, with the flapping axis as origin (fig. 1)
ξ	abscissa along axis OX
y	ordinate of a point of the elastically deflected blade along an axis OY normal to axis OX

[1]The "mode 0" is that which corresponds, for a hinged blade, to a vibration without bending, that is, to flapping β. The mode 1 for such a blade is then that which corresponds to a vibration of one node.

[2]This damping term has, obviously, no significance when there is separation of flow, because beyond angle of separation the normal lift coefficient is practically independent of the angle of attack.

a	distance of the flapping hinge from the axis of rotation of the rotor
β	flapping angle of the rigid blade
m'	mass of the blade per unit length at a point under consideration
l	blade chord
E	Young's modulus of the blade material
I	moment of inertia of a normal section of the blade
ω	angular velocity of the blade
t	time
$\psi = \omega t$	azimuth angle of the blade at time t
ψ_n	phase difference of the azimuth angle due to damping
V	forward speed of helicopter
v	resultant velocity of the air on a section of the rigid blade
V_N	normal component of velocity v
V_T	tangential component of velocity v
θ	angle of attack of a blade section
φ	angle of the resultant velocity v with the normal plane
$\Delta\varphi$	angle of velocity v with the resultant velocity on the deflected blade $\left(\frac{\dot{y} \cos \varphi}{v}\right)$
α	angle of the normal plane with the forward velocity
μ	ratio of advance $\left(\frac{V \cos \alpha}{\omega R}\right)$
ρ	air density
dC_z/di	lift-curve slope of the profile
g	acceleration of gravity

s static moment with respect to the axis of rotation of the part of the blade located beyond the abscissa x $\left(\int_x^R \dot{m}'(a + \xi)d\xi\right)$

Φ coefficient of damping due to deflection of the blade. (See text.)

η_i natural deflection function of order i

$\nu_{1,0}$ natural frequency of the order i of the nonrotating blade

$\nu_{i,\omega}$ natural frequency of the order i of the blade caused by rotation with angular velocity ω

$\left.\begin{matrix} g_i \\ h_i \end{matrix}\right\}$ auxiliary functions, dependent on time only

n order of a harmonic in Fourier series

na, nb subscripts of terms in cosine and sine of a Fourier series

THE EQUATION OF DEFLECTION OF THE BLADE

Let us consider the forces which act at S on a blade element of span dx in the plane YOX. (See fig. 1.) These forces are:

(1) The elementary lift[3] $dF = \frac{\rho}{2} l \frac{dC_z}{di}\left(\theta V_T^2 - V_N V_T\right)dx$

(2) A corrective term of the damping of the lift force due to the flexural elastic deformation of the blade. This term is of the form

$$-K\dot{y}\ dx$$

(3) The weight $dp = m'g\ dx$, the components of which along OX and OY are, respectively

$$m'g \sin \beta\ dx \text{ (negligible)}$$

$$m'g \cos \beta\ dx \simeq m'g\ dx$$

[3]This elementary lift already includes, according to the definition of V_N, a term due to the aerodynamic damping of the flapping rigid blade.

(4) The centrifugal force of rotation $m'\omega^2 r\,dx$, the components of which along OX and OY are, respectively

$$m'\omega^2(a + x\cos\beta)dx \simeq m'\omega^2 r\,dx$$

and

$$m'\omega^2(a + x\sin\beta)dx \simeq m'\omega^2 r\beta\,dx$$

(5) The force of inertia of flapping: $-m'x\ddot{\beta}\,dx$

(6) The force of inertia of deflection: $-m'\ddot{y}\,dx$

Strictly speaking, the following forces should also be included:

(7) The centrifugal force of flapping: $m'x\dot{\beta}^2\,dx$

(8) The Coriolis force due to the simultaneous action of blade deflection and flapping motion:

$$-2\dot{\beta}\dot{y}m'\,dx$$

however, we shall disregard them in relation to the centrifugal force of rotation.

For the blade element $d\xi$ to be in equilibrium, it is necessary to add to these forces the unknown actions of the adjacent elements on the element under consideration.

Calculation of $K\dot{y}$

$$K\dot{y} = \frac{\rho}{2}\, l\, \frac{dC_z}{di}\, v^2\Delta\varphi$$

$$v = \frac{V_T}{\cos\varphi} = \frac{\omega r + V\cdot\cos\alpha\,\sin\psi}{\cos\varphi}$$

$$\Delta\varphi = \frac{\dot{y}\cos\varphi}{v} \text{ (See fig. 2.)}$$

$$= \frac{\dot{y}\cos^2\varphi}{\omega r + V\cos\alpha\,\sin\psi}$$

It becomes

$$K = \frac{\rho}{2} l \frac{dC_z}{di} (\omega r + V \cos \alpha \sin \psi)$$

Since the calculation can be made only when the coefficient of the damping term due to the deflection is independent of time, the periodic part in $V \sin \psi$ will be disregarded. This reduces the problem to the corresponding mean speed of the air with respect to the blade element.

Hence

$$K = \frac{\rho}{2} l \frac{dC_z}{di} \omega r$$

The differential equation of the blade then reads

$$\frac{d^2}{dx^2}\left(EI \frac{d^2y}{dx^2}\right) - \omega^2 \frac{d}{dx}\left[\frac{dy}{dx}\int_x^R m'(a + \xi)d\xi\right] + m'\ddot{y} + K\dot{y}$$

$$= \frac{dF}{dx} - m'g - m'x\ddot{\beta} - m'\omega^2 r\beta \tag{1}$$

The dots signify the derivatives with respect to time.

This equation of the partial derivatives must be completed by four limiting conditions which define the integration constants, namely,

$$y = 0 \quad \text{for} \quad x = 0$$

$$EI \frac{d^2y}{dx^2} = 0 \quad \text{for} \quad x = 0 \quad \text{and for} \quad x = R - a$$

$$\frac{d}{dx}\left(EI \frac{d^2y}{dx^2}\right) = 0 \quad \text{for} \quad x = R - a$$

METHOD OF SOLUTION

The foregoing deflection equation is composed of a first member with terms dependent on the deflection, and of a second member which is independent of deflection. This second member comprises the aerodynamic forces, the weight and the forces of inertia of rotation, and flapping acting on the rigid blade. These forces are easily computed for each point of the blade and for each one of its azimuth positions after the equation of flapping β of the rigid blade has been solved. We shall waive their calculation and identify the second member of the preceding equation by the function $F'_d(x,t)$. We further put

$$\int_x^R m'(a + \xi)d\xi = s$$

so that the preceding equation reads

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2y}{dx^2}\right) - \omega^2\,\frac{d}{dx}\left(s\,\frac{dy}{dx}\right) + m'\ddot{y} + K\dot{y} = F'_d(x,t) \tag{2}$$

This is an equation of partial derivatives with second member, representing the forced vibrations of the blade with damping.

To resolve it, we introduce the natural functions of the deflections of the blade, that is, the vibrations obtained by solving the foregoing equation of the partial derivatives above without second member.

Consider first, for simplicity, a state of forced vibrations without damping arising from the deflections of the blade. (In other words, the term $K\dot{y}$ is disregarded.)

When we consider the moment, M_{rig}, of the forces exerted on the assumedly rigid blade, this moment is, at a point of the blade, a function of time only and can therefore be developed in series of periodic functions of $\psi = \omega t$

$$M_{rig} = M_0 + M_a \cos\psi + M_b \sin\psi + M_{2a} \cos 2\psi + M_{2b} \sin 2\psi$$

$$+ M_{3a} \cos 3\psi + M_{3b} \sin 3\psi + \ldots$$

This moment is none other than the moment of the distributed outside forces F'_d appearing in the second member of the preceding equation.

Let us resolve these forces at each instant in series of distributions such that they each impart to the blade a deflection taking the form of a natural mode of deflection of the corresponding order.

It can be shown that an arbitrary deflection of the blade may always be resolved in series of natural functions by reason of the relations of orthogonality existing between the natural functions of continuous beams, whatever their limiting conditions. (Physically, it means that the different natural vibrations act independent of each other without mutual interactions.)

The proposed resolution has the form

$$F'_d = g_0 m' \eta_0 + g_1 m' \eta_1 + g_2 m' \eta_2 + \ldots \tag{3}$$

where g_i is a function of time only and η_i is the natural function of the deflection of the blade of the order i.

The deflection of the order i of the blade affects then, at each instant, the form of the function η_i, that is, it will be given by

$$y_i = h_i \eta_i$$

where h_i is a function of the time only. The total deflection is, by virtue of the relations of orthogonality invoked above, obtained by superposition of the various natural deflections of the blade vibrating at the corresponding natural frequencies, with amplitudes and phase differences defined by the function h_i, which, itself, is obtained by putting the expression $y_i = h_i \eta_i$ in the equation of the partial derivatives.

Hence, for a frequency of the order i

$$h_i \frac{d^2}{dx^2}\left(EI \frac{d^2\eta_i}{dx^2}\right) - h_i \omega^2 \frac{d}{dx}\left(s \frac{d\eta_i}{dx}\right) + \ddot{h}_i m' \eta_i = g_i m' \eta_i \tag{4}$$

Now it is known that an equation of the form

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2y}{dx^2}\right) - \omega^2 \frac{d}{dx}\left(s\,\frac{dy}{dx}\right) + m'\ddot{y} = 0$$

representing the natural vibrations of a freely vibrating blade and involved in rotation with an angular velocity ω permits an infinite number of solutions of the form $\eta \sin \nu t$, satisfying

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2\eta}{dx^2}\right) - \omega^2 \frac{d}{dx}\left(s\,\frac{d\eta}{dx}\right) = \nu^2 m'\eta$$

and the limiting conditions. Every solution η_i is the natural function of order i corresponding to the natural frequency $\nu_{i,\omega}$.

Therefore, when η_i is a natural function, it simultaneously satisfies the equation (4) and the equation

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2\eta_i}{dx^2}\right) - \omega^2 \frac{d}{dx}\left(s\,\frac{d\eta_i}{dx}\right) = \nu^2_{i,\omega} m'\eta_i$$

where $\nu_{i,\omega}$ is the corresponding natural frequency of the blade actuated with a speed of rotation ω.

Hence, after simplification

$$h_i \nu^2_{i,\omega} + \ddot{h}_i = g_i$$

Since the functions h_i and g_i are periodic with respect to ψ, they can be developed in harmonic series

$$h_i = h_{i,0} + h_{i,a} \cos \psi + h_{i,b} \sin \psi + h_{i,2a} \cos 2\psi$$

$$+ h_{i,2b} \sin 2\psi + h_{i,3a} \cos 3\psi + h_{i,3b} \sin 3\psi \ldots$$

$$g_i = g_{i,0} + g_{i,a} \cos \psi + g_{i,b} \sin \psi + g_{i,2a} \cos 2\psi$$

$$+ g_{i,2b} \sin 2\psi + g_{i,3a} \cos 3\psi + g_{i,3b} \sin 3\psi \ldots$$

The differential equation

$$\sum_{i=0}^{i=\infty}\left[h_i \frac{d^2}{dx^2}\left(EI \frac{d^2\eta_i}{dx^2}\right) - h_i\omega^2 \frac{d^2}{dx}\left(s \frac{d\eta_i}{dx}\right) + \ddot{h}_i m'\eta_i\right] = \sum_{i=0}^{i=\infty} g_i m'\eta_i$$

therefore resolves itself by identification of the coefficients, and limiting it to the third harmonic, it reduces to the system

$$
\begin{aligned}
&h_{i,0}\nu^2{}_{i,\omega} = g_{i,0}\\
&h_{i,a}\left(\nu^2{}_{i,\omega} - \omega^2\right) = g_{i,a}\\
&h_{i,b}\left(\nu^2{}_{i,\omega} - \omega^2\right) = g_{i,b}\\
&h_{i,2a}\left(\nu^2{}_{i,\omega} - 4\omega^2\right) = g_{i,2a}\\
&h_{i,2b}\left(\nu^2{}_{i,\omega} - 4\omega^2\right) = g_{i,2b}\\
&h_{i,3a}\left(\nu^2{}_{i,\omega} - 9\omega^2\right) = g_{i,3a}\\
&h_{i,3b}\left(\nu^2{}_{i,\omega} - 9\omega^2\right) = g_{i,3b}
\end{aligned}
\qquad (5)
$$

Each function g_i is well defined.

To determine it, simply multiply the two members of equation (3) by η_i and integrate over the blade.

Owing to the conditions of orthogonality

$$\int_0^R m'\eta_i\eta_j \, dx = 0 \qquad i \neq j$$

it leaves

$$\int_0^R F'_d \eta_i \, dx = g_i \int_0^R m' \eta_i{}^2 dx$$

hence

$$g_i = \frac{\int_0^R F'_d \eta_i \, dx}{\int_0^R m' \eta_i{}^2 dx} \tag{6}$$

The bending moment exerted on the elastically deformable blade is computed next.

The elastic deformation due to a single harmonic of the outside forces, such as

$$\sum_{i=1}^{i=\infty} \left[g_{i,na} m' \eta_i \cos n\psi + g_{i,nb} m' \eta_i \sin n\psi \right]$$

is, as shown previously,

$$\sum_{i=1}^{i=\infty} y_{i,n} \begin{Bmatrix} a \\ b \end{Bmatrix} = \sum_{i=1}^{i=\infty} h_{i,n} \begin{Bmatrix} a \\ b \end{Bmatrix} \eta_i$$

On replacing the terms h_i by their values obtained from equation (5), the corresponding bending moment reads

$$M_{elast.\ n} \begin{Bmatrix} a \\ b \end{Bmatrix} = \sum_{i=1}^{i=\infty} \left[\frac{g_{i,n} \begin{Bmatrix} a \\ b \end{Bmatrix}}{\nu^2{}_{i,\omega} - n^2 \omega^2} EI \frac{d^2 \eta_i}{dx^2} \begin{Bmatrix} \cos n\psi \\ \sin n\psi \end{Bmatrix} \right] \tag{7}$$

On making the exact calculation of the natural functions η_i, it is found that they vary very little with ω and that a natural function of the order i, for $\omega = 0$, can be compared with the same natural function for the normal speed ω.

$$\eta_{i,\omega} \simeq \eta_{i,0} \simeq \eta_i$$

This simplification is not exact, but the resulting error is small (less than 3 percent for the first natural function in the case of the blade cited in the example hereinafter), being of the same order as those committed in the distribution of the masses and the flexural stiffness of the blade.

In this case, the natural function η_i satisfies both

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2\eta_i}{dx^2}\right) - \omega^2\,\frac{d}{dx}\left(s\,\frac{d\eta_i}{dx}\right) = \nu^2{}_{i,\omega}m'\eta_i$$

and

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2\eta_i}{dx^2}\right) = \nu^2{}_{i,0}m'\eta_i$$

Hence

$$EI\,\frac{d^2\eta_i}{dx^2} = \nu^2{}_{i,0}\iint_x^R m'\eta_i\ dx\ d\xi$$

On replacing $EI\,\dfrac{d^2\eta_i}{dx^2}$ by its value in (7), the coefficient of the harmonic n of the elastic moment which, in fact, bends the flexible blade, is given by the expression

$$M_{elast.\ n}\begin{Bmatrix}a\\b\end{Bmatrix} = \sum_{i=1}^{i=\infty}\left(\frac{\nu^2{}_{i,0}}{\nu^2{}_{i,\omega} - n^2\omega^2}\,g_{i,n}\begin{Bmatrix}a\\b\end{Bmatrix}\iint_x^R m'\eta_i\ dx\ d\xi\right) \tag{8}$$

The value g_i is computed by (6) for the values of ψ for which F'_d is given. The development of g_i in Fourier series defines the coefficient

$$g_{i,n}\begin{cases}a\\b\end{cases}$$

The natural functions η_i and the integrals $\iint m'\eta_i\,dx\,d\xi$ can be computed by a classical method such as the iteration method (Stodola), or the Galerkin method.

There is no occasion to be preoccupied with normalizing the natural functions. For the amplitude of η_i, any convenient scale is suitable; the effect of the scale disappears later in the product

$$g_i\iint_x^R m'\eta_i\,dx\,d\xi$$

The calculation for $\omega = 0$ is made while remembering the previous statement that the deflection is practically unmodified by rotation.

If the iterative method is used for computing the natural functions of the deflection, the natural frequencies of the blade not rotating and of the blade rotating at angular velocity ω can be computed by applying Rayleigh's method to the obtained natural deflection. This method affords rigorous solutions, converges rapidly, and avoids the solution of n equations with n unknowns to which the Galerkin method leads.

> Note: The bending moments could also be computed by direct application of equation (7). This method is predicated on the exact knowledge of $d^2\eta_i/dx^2$ which prohibits the use of the approximation $\eta_{i,\omega} = \eta_{i,0}$ because a slight error in a function can cause a substantial error in its second derivative. The calculations of the natural deflections are quite complicated. On the other hand, the function y being defined by dots, it is not possible to derive it directly to obtain d^2y/dx^2.

SIMPLIFICATION OF THE CALCULATIONS

Numerous calculations made on blades of various helicopters with different plan forms and distributions of masses and different amounts of rigidity have shown that, in certain cases - blades of moderately conical shape, little twist, and lightly loaded at the tip - the natural functions, other than the first, exert little influence on the maximum bending moments exerted on the blade, and consequently on the maximum alternating fatigue stresses to which the blade is subjected. This is because the distributions of the outside forces (curves $F'_d = f(\overline{r})$ for the various ψ) for such blades represent the behavior of the first natural distribution (curve $m'\eta = f(\overline{r})$) in a satisfactory manner and also because the natural functions of higher orders present all the "loops" and "nodes" in continuously increasing number, matching poorly the greater deflection which the maximum moments produce.

It is only in cases of small deflections that the "parasite remainder" of the higher frequencies, arising from the fact that the distributions of the outside forces never have a curve exactly identical with the first natural distribution, can play a significant part.

When the blades have pronounced camber and twist and are lightly loaded at the tips, the forces of inertia can become more important than the aerodynamic forces at the blade tip in the entire sector of the swept disk corresponding to the advancing blade. The distributions of the outside forces $F'_d = f(\overline{r})$ can assume, therefore, the curves approached by the second natural distribution (curve $m'\eta_2$) for an entire series of ψ, and, in that case, the second and sometimes the third natural functions must be taken into consideration in the calculation of the maximum moments exerted at the blade.

Examination of the curves $F'_d = f(\overline{r})$ permits one to determine, with a little practice, when resolution of the outside forces can be limited to the first natural distribution and the deflection of the blade to that corresponding to the first natural function.

In the latter case, the calculations are considerably simplified.

The distribution of the outside forces is reduced to

$$F'_d = g_1 m' \eta_1$$

The bending moment on the elastic blade becomes

$$M_{elast.\ n}\begin{cases}a\\b\end{cases} = \frac{\nu^2_{1,0}}{\nu^2_{1,\omega} - n^2\omega^2}\, g_{1,n}\begin{cases}a\\b\end{cases}\iint_x^R m'\eta_1\, dx\, d\xi \qquad (9)$$

On comparing this bending moment with the moment of the outside force distribution F'_d exerted on the assumedly infinitely rigid blade, we get

$$M_{rig} = \iint_x^R F'_d\, dx\, d\xi = g_1 \iint_x^R m'\eta_1\, dx\, d\xi \qquad (10)$$

It is seen that the bending moment on the elastic blade is obtained by multiplying the coefficients of the harmonics of the same order of moment on the rigid blade by a factor

$$A_n = \frac{\nu^2_{1,0}}{\nu^2_{1,\omega} - n^2\omega^2} \qquad (11)$$

which is constant over the blade for a given harmonic.

It is no longer necessary to calculate the natural functions η_1 and the natural frequencies $\nu_{1,0}$ and $\nu_{1,\omega}$. They can be readily and closely approximated by the Rayleigh method applied to a curve representing approximately the deflection of the blade while still satisfying the limiting conditions rigorously.

As regards the various harmonics to be kept for the moments on the rigid blade, it seems that no advantages are gained by going beyond the third, which is already relatively small.

The bending moment, at a point on the abscissa x, on the rigid blade is then given by the expression

$$M_{rig} = M_0 + M_a \cos\psi + M_b \sin\psi + M_{2a}\cos 2\psi +$$

$$M_{2b}\sin 2\psi + M_{3a}\cos 3\psi + M_{3b}\sin 3\psi$$

The effective moment bending the actual elastic blade at the same point is obtained by the simple relation

$$M_{elast.} = \frac{\nu^2_{1,0}}{\nu^2_{1,\omega}} M_0 + \frac{\nu^2_{1,0}}{\nu^2_{1,\omega} - \omega^2} \left(M_a \cos \psi + M_b \sin \psi\right) +$$

$$\frac{\nu^2_{1,0}}{\nu^2_{1,\omega} - 4\omega^2} \left(M_{2a} \cos 2\psi + M_{2b} \sin 2\psi\right) +$$

$$\frac{\nu^2_{1,0}}{\nu^2_{1,\omega} - 9\omega^2} \left(M_{3a} \cos 3\psi + M_{3b} \sin 3\psi\right)$$

CALCULATION INCLUDING AERODYNAMIC DAMPING DUE TO DEFLECTION - GENERAL METHOD

In this case, the term $K\dot{y}$ disregarded in equation (2) must be included.

It has been shown that

$$K = \frac{\rho}{2} l \frac{dC_z}{di} \omega r$$

The term $\frac{\rho}{2} l \frac{dC_z}{di} r$ is homogeneous to a distributed mass.

Putting

$$\Phi(x) = \frac{\rho}{2} \frac{l r}{m'} \frac{dC_z}{di}$$

gives

$$K = \Phi_{(x)} m' \omega$$

where $\Phi_{(x)}$ is a dimensionless coefficient. Since, as a rule, $\frac{\rho}{2} l \frac{dC_z}{di} r$ is not proportional to the distributed mass m', the coefficient $\Phi_{(x)}$ is a function of x.

Having already permitted one approximation for K by including only the corresponding mean speed on a blade element, another simplification is effected by bringing only the mean value of $\Phi_{(x)}$ along the span of the blade into the first member of the equation of motion.

$$\Phi = \frac{1}{R - a} \int_0^R \frac{\rho}{2} \frac{l r}{m'} \frac{dC_z}{di} dx$$

Accordingly, equation (2) reads

$$\frac{d^2}{dx^2}\left(EI \frac{d^2 y}{dx^2}\right) - \omega^2 \frac{d}{dx}\left(s \frac{dy}{dx}\right) + m'\ddot{y} + \Phi m' \omega \dot{y} = F'_{d(x,r)} \qquad (12)$$

With the symbols already employed, we get

$$h_i \nu^2_{i,\omega} + \ddot{h}_i + \dot{h} \Phi \omega = g_i$$

The foregoing system of equations (5) is therefore replaced by another of the form

$$\left.\begin{aligned} h_{i,na}\left(\nu^2_{i,\omega} - n^2\omega^2\right) + h_{i,nb} n\omega^2 \Phi &= g_{i,na} \\ h_{i,nb}\left(\nu^2_{i,\omega} - n^2\omega^2\right) - h_{i,na} n\omega^2 \Phi &= g_{i,nb} \end{aligned}\right\} \qquad (13)$$

whence

$$h_{i,na} = \frac{g_{i,na}\left(\nu^2_{i,\omega} - n^2\omega^2\right) - g_{i,nb}n\omega^2\Phi}{\left(\nu^2_{i,\omega} - n^2\omega^2\right)^2 + n^2\omega^4\Phi^2}$$

$$h_{i,nb} = \frac{g_{i,nb}\left(\nu^2_{i,\omega} - n^2\omega^2\right) + g_{i,na}n\omega^2\Phi}{\left(\nu^2_{i,\omega} - n^2\omega^2\right)^2 + n^2\omega^4\Phi^2}$$

Putting

$$\tan \psi_{i,n} = \frac{n\omega^2\Phi}{\nu^2_{i,\omega} - n^2\omega^2}$$

therefore

$$\frac{n\omega^2\Phi}{\sqrt{\left(\nu^2_{i,\omega} - n^2\omega^2\right)^2 + n^2\omega^4\Phi^2}} = \sin \psi_{i,n}$$

$$\frac{\nu^2_{i,\omega} - n^2\omega^2}{\sqrt{\left(\nu^2_{i,\omega} - n^2\omega^2\right)^2 + n^2\omega^4\Phi^2}} = \cos \psi_{i,n}$$

Consequently

$$h_{i,na} = \frac{g_{i,na} \cos \psi_{i,n} - g_{i,nb} \sin \psi_{i,n}}{\sqrt{\left(\nu^2{}_{i,\omega} - n^2\omega^2\right)^2 + n^2\omega^4\phi^2}} \tag{14a}$$

$$h_{i,nb} = \frac{g_{i,na} \sin \psi_{i,n} + g_{i,nb} \cos \psi_{i,n}}{\sqrt{\left(\nu^2{}_{i,\omega} - n^2\omega^2\right)^2 + n^2\omega^4\phi^2}} \tag{14b}$$

The calculation is then carried out in the same way as without damping; the deflection due to a single harmonic n of the outside forces being always

$$y_n = \sum_{i=1}^{i=\infty} \left(h_{i,na}\eta_i \cos n\psi + h_{i,nb}\eta_i \sin n\psi\right)$$

Replacing h_i by its value gives

$$y_n = \sum_{i=1}^{i=\infty} \left\{ \frac{\eta_i}{\sqrt{\left(\nu^2{}_{i,\omega} - n^2\omega^2\right)^2 + n^2\omega^4\phi^2}} \left[\left(g_{i,na} \cos \psi_{i,n} - g_{i,nb} \sin \psi_{i,n}\right)\cos n\psi + \left(g_{i,na} \sin \psi_{i,n} + g_{i,nb} \cos \psi_{i,n}\right)\sin n\psi\right] \right\}$$

Hence, as before, the term of the harmonic n of the moment which, in fact, bends the flexible blade

$$M_{elast.\ n} = \sum_{i=1}^{i=\infty} \left\{ \frac{\nu^2{}_{i,0} \iint_x^R m'\eta_i \, dx \, d\xi}{\sqrt{\left(\nu^2{}_{i,\omega} - n^2\omega^2\right)^2 + n^2\omega^4\phi^2}} \left[\left(g_{i,na} \cos \psi_{i,n} - g_{i,nb} \sin \psi_{i,n}\right)\cos n\psi + \left(g_{i,na} \sin \psi_{i,n} + g_{i,nb} \cos \psi_{i,n}\right)\sin n\psi\right]\right.$$

or also

$$M_{elast.\ n} = \sum_{i=1}^{i=\infty} \left\{ \frac{\nu^2_{i,0} \iint_x^R m' \eta_i \; dx \; d\xi}{\sqrt{\left(\nu^2_{i,\omega} - n^2\omega^2\right)^2 + n^2\omega^4\Phi^2}} \left[g_{i,na} \cos(n\psi - \psi_{i,n}) + g_{i,nb} \sin(n\psi - \psi_{i,n})\right] \right\} \tag{15}$$

with

$$\tan \psi_{i,n} = \frac{n\omega^2\Phi}{\nu^2_{i,\omega} - n^2\omega^2}$$

If, as previously, the distribution of the outside forces is limited to the first distribution and the deflection of the blade to that corresponding to the first natural function, the following rule results: The effective bending moments acting on an elastic blade, with damping, are obtained from the bending moments acting on the assumedly rigid blade, each harmonic of which is modified as follows:

(1) Its amplitude is multiplied by a coefficient equal to

$$\frac{\nu^2_{1,0}}{\sqrt{\left(\nu^2_{i,\omega} - n^2\omega^2\right)^2 + n^2\Phi^2\omega^4}}$$

(2) There is a forward phase difference of ψ_n, so that

$$\tan \psi_n = \frac{n\Phi\omega^2}{\nu^2_{1,\omega} - n^2\omega^2}$$

PRACTICAL PROCEDURE OF CALCULATION IN THE CASE OF THE GENERAL METHOD

(The Double Resolution of the Outside Forces Being Limited to the Third Natural Distribution and to the Third Harmonic in ψ)

The first step is to determine the natural functions of the blade η_1, η_2, and $\eta_3 = f(\overline{r})$, as well as the corresponding natural frequencies $\nu_{1,0}$, $\nu_{2,0}$, and $\nu_{3,0}$ of the nonrotating blade and the $\nu_{1,\omega}$, $\nu_{2,\omega}$, and $\nu_{3,\omega}$ of the blade rotating at angular velocity ω.

Next, it is necessary to evaluate the quantities

$$\int_0^R m'{\eta_1}^2 dx, \quad \int_0^R m'{\eta_2}^2 dx, \quad \text{and} \quad \int_0^R m'{\eta_3}^2 dx$$

and then plot the curves

$$\iint_x^R m'\eta_1 \, dx \, d\xi, \quad \iint_x^R m'\eta_2 \, dx \, d\xi, \quad \text{and} \quad \iint_x^R m'\eta_3 \, dx \, d\xi$$

against $\overline{r}$.

Then determine the outside forces on the rigid blade $F'_d = f(\overline{r})$ for different ψ (eight positions spaced 45° apart must be explored).

Next evaluate for each position

$$\int_0^R F'_d\eta_1 \, dx, \quad \int_0^R F'_d\eta_2 \, dx, \quad \text{and} \quad \int_0^R F'_d\eta_3 \, dx$$

Thence one obtains

$$g_1 = \frac{\int_0^R F'_d\eta_1 \, dx}{\int_0^R m'{\eta_1}^2 dx}, \quad g_2 = \frac{\int_0^R F'_d\eta_2 \, dx}{\int_0^R m'{\eta_2}^2 dx}, \quad g_3 = \frac{\int_0^R F'_d\eta_3 \, dx}{\int_0^R m'{\eta_3}^2 dx}$$

for each position.

Develop g_1, g_2, g_3 in Fourier series (by the Runge method, for example) and stop with the third harmonic of ψ:

whence

$$\begin{matrix} g_{1,0} & g_{1,a} & g_{1,b} \cdots g_{1,3b} \\ g_{2,0} & g_{2,a} & g_{2,b} \cdots g_{2,3b} \\ g_{3,0} & g_{3,a} & g_{3,b} \cdots g_{3,3b} \end{matrix}$$

Calculate the bending moments on the blade for different stations $x = K.R$ by the formula

$$M_x = \sum_{i=1}^{i=3} M_{i,x}$$

with

$$M_{i,x} = \iint_x^R m'\eta_i \, dx \, d\xi \Big[A_{i,0} + A_{i,1} \cos\left(\psi - \psi_{i,1} - \varphi_{i,1}\right) +$$

$$A_{i,2} \cos\left(2\psi - \psi_{i,2} - \varphi_{i,2}\right) + A_{i,3} \cos\left(3\psi - \psi_{i,3} - \varphi_{i,3}\right)\Big]$$

where

$$A_{i,n} = \frac{\nu^2{}_{i,0}\sqrt{g^2{}_{i,na} + g^2{}_{i,nb}}}{\sqrt{\left(\nu^2{}_{i,\omega} - n^2\omega^2\right)^2 + n^2\phi^2\omega^4}}, \text{ always positive}$$

$$A_{i,0} = \frac{\nu^2{}_{i,0}}{\nu^2{}_{i,\omega}} g_{i,0}$$

$$\tan \psi_{i,n} = \frac{n\phi\omega^2}{\nu^2{}_{i,\omega} - n^2\omega^2}$$

$$\tan \varphi_{i,n} = \frac{g_{i,nb}}{g_{i,na}}$$

To determine quadrants of ψ, φ: For $\psi_{i,n}$, $\dfrac{\nu^2{}_{i,\omega} - n^2\omega^2}{\cos \psi_{i,n}} > 0$ and for $\varphi_{i,n}$, $g_{i,na}/\cos \varphi_{i,n} > 0$.

PRACTICAL EXAMPLE

Hereinafter follows an application of this method to the calculation of the bending moments exerted on the blade of the helicopter N.C. 2001 in forward flight, with $\mu = 0.43$.

The mechanical characteristics of the blade, which has a radius of R = 6.85m, are given in figure 3. Figures 4 and 5 give the natural functions η and the blade distributions $m'\eta$. Figure 6 gives the natural frequencies of the nonrotating blade and of the blade rotating at angular velocity ω for the different modes of vibration. Figure 7 gives the distribution of the aerodynamic forces, and figure 8 the distribution of the total outside forces F'_d on the rigid blade for several azimuth positions of the latter.

According to figure 8, the distributions of the outside forces for the positions $\varphi = 45^\circ$, 90°, 135°, and 180° have clearly the shape of the second natural distribution $m'\eta_2$, which explains, as will be seen in figures 9 to 12, the importance of the bending moments computed with the second natural function included.

Figure 13 shows the bending moment distribution for several radii plotted again ψ, and figure 14 the enveloping curve of the maximum bending moments exerted on the blade. Figure 14 also shows the maximum bending moment curve acting on the blade of the S.E. 3000 helicopter (R = 6m, $\mu = 0.41$), whose mechanical characteristics are shown also in figure 3. It is seen that for this blade, which presents an average camber and is lightly loaded at the tip, the error made in the maximum bending moment by limiting the distributions of the outside forces to the first distribution does not exceed 6 percent, which justifies the simplification of the calculation indicated previously.

For the blade of the NC.2001, the maximum bending moment is severely subjected to the influence of the second distribution, but, by way of compensation, it is practically clear of that of the third distribution (except at the tip).

However, it should be noted that the latter assumes a significant part in the evaluation of the negative maximum bending moment, expressed

by M_{min} (see figs. 10 and 13), and consequently in the appraisal of the maximum alternating fatigue, defined by

$$f_{max} = \pm \frac{M_{max} - M_{min}}{2W}$$

where $W = \frac{I}{V}$ is the resistant modulus of the particular section.

Translated by J. Vanier
National Advisory Committee
for Aeronautics

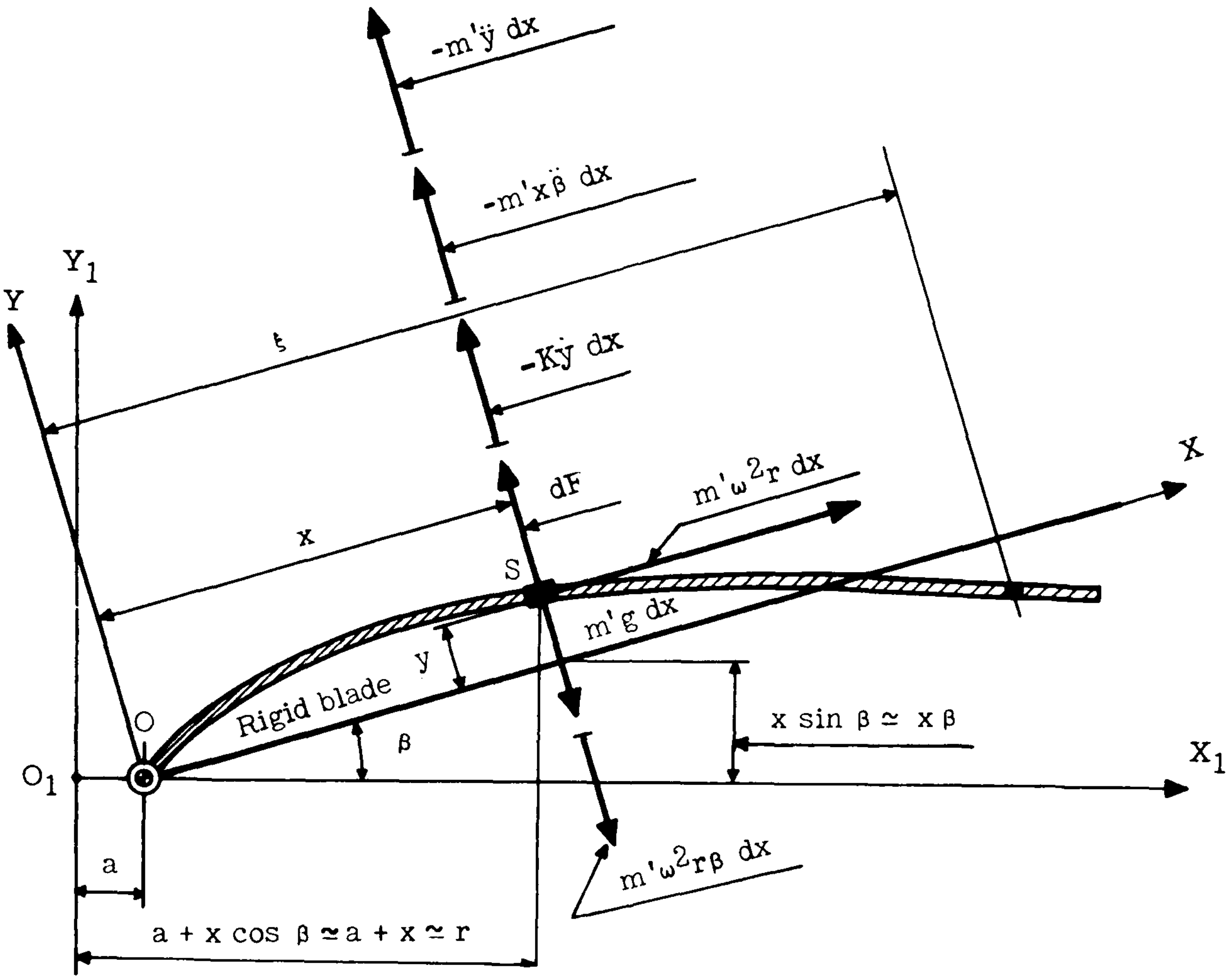

Figure 1.

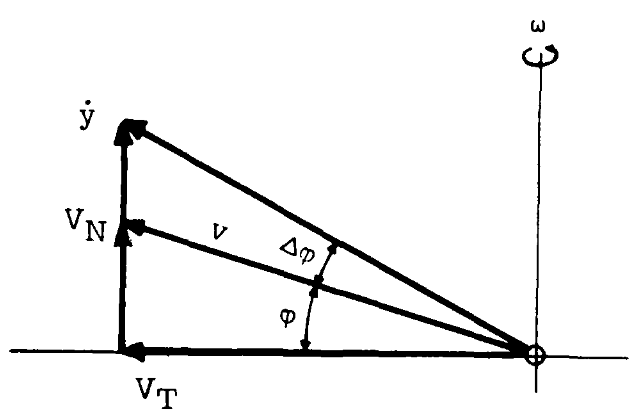

Figure 2.

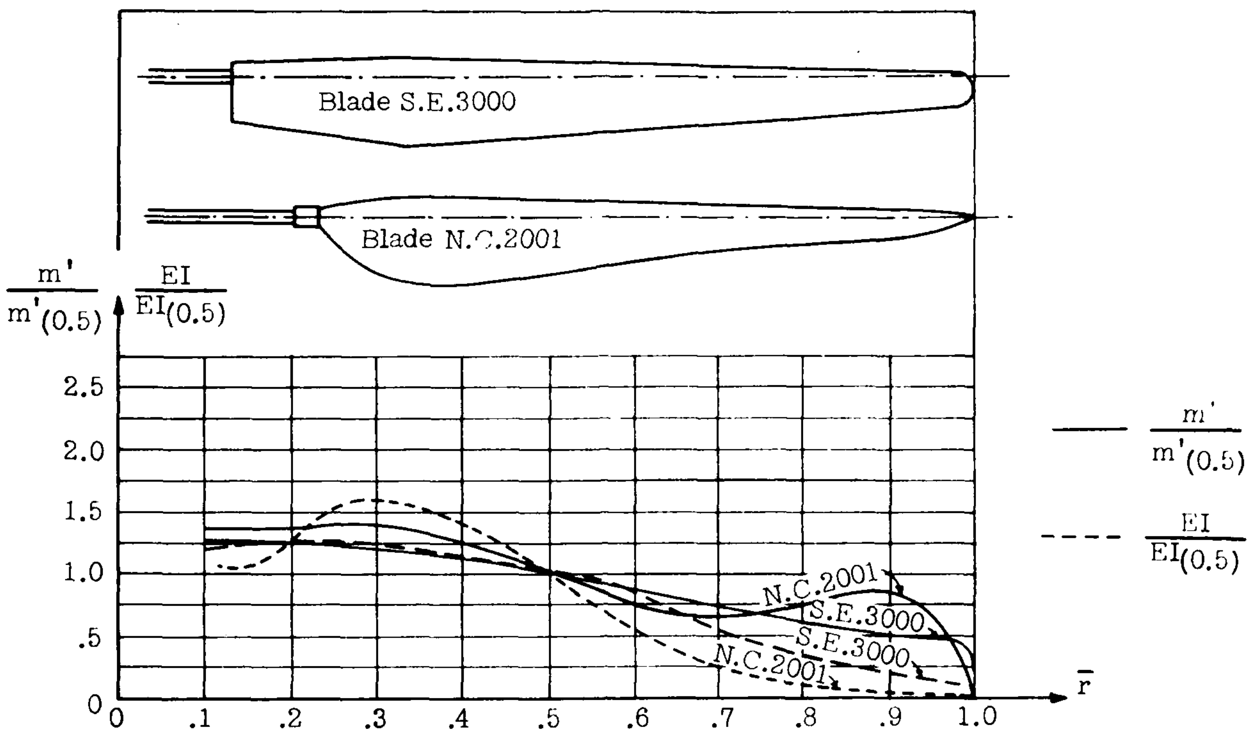

Figure 3.- Plan forms and distribution of mass and rigidity of the S.E.3000 and N.C.2001 helicopter rotor blades.

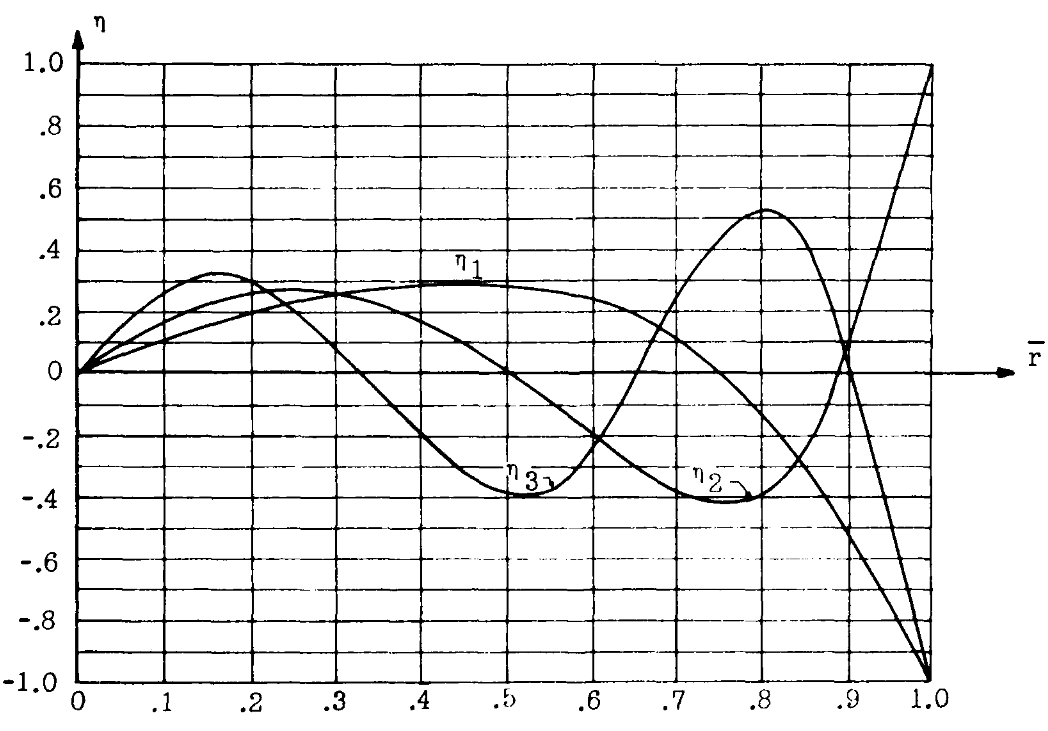

Figure 4.- N.C.2001 blade - natural elastic deflection curves of the blade.

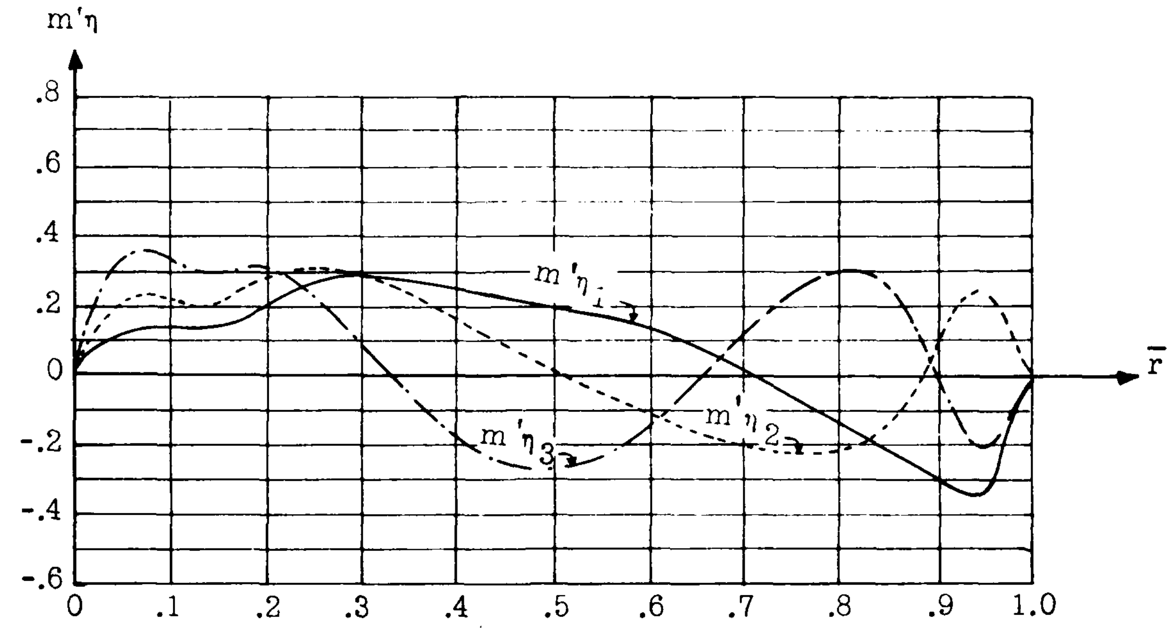

Figure 5.- N.C.2001 blade - natural distributions $m'\nu^2\eta = \frac{d^2}{dx^2}\left(EI\frac{d^2\eta}{dx^2}\right)$ for $\nu = 1$.

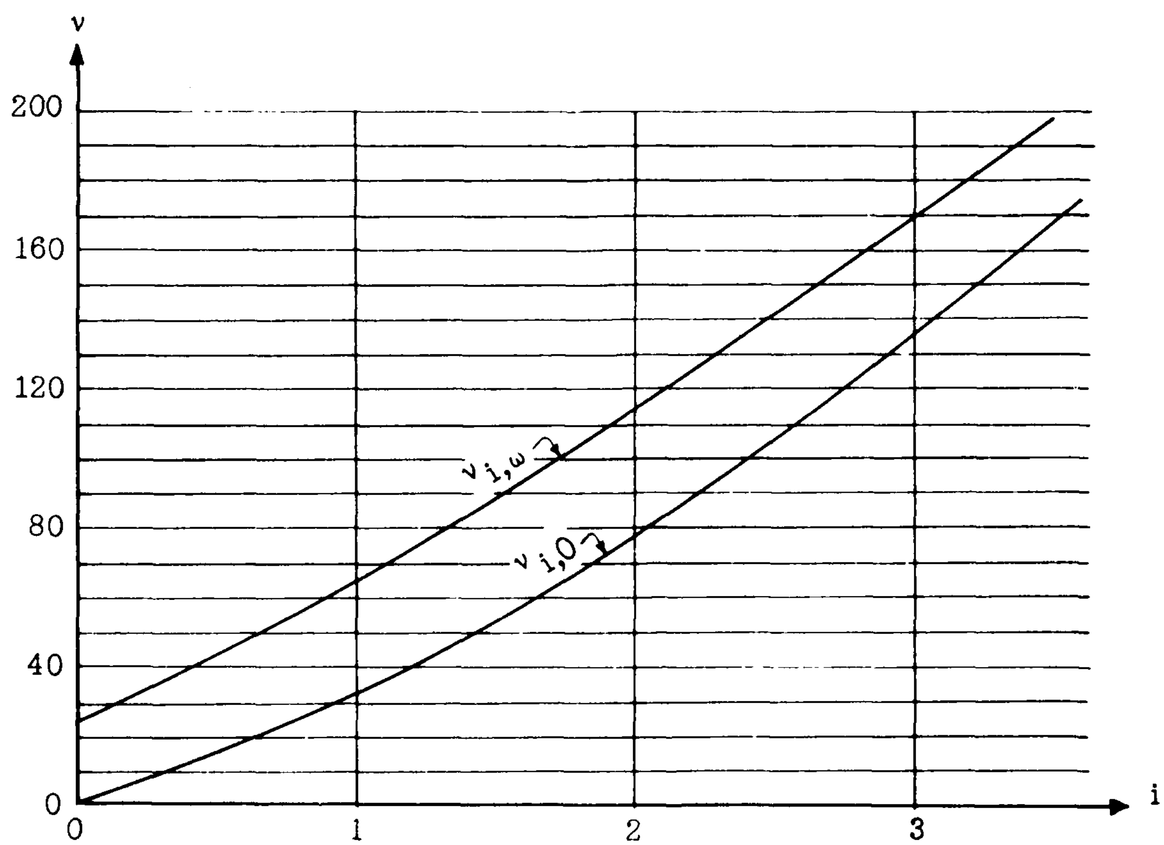

Figure 6.- Natural frequencies of the N.C.2001 blade.

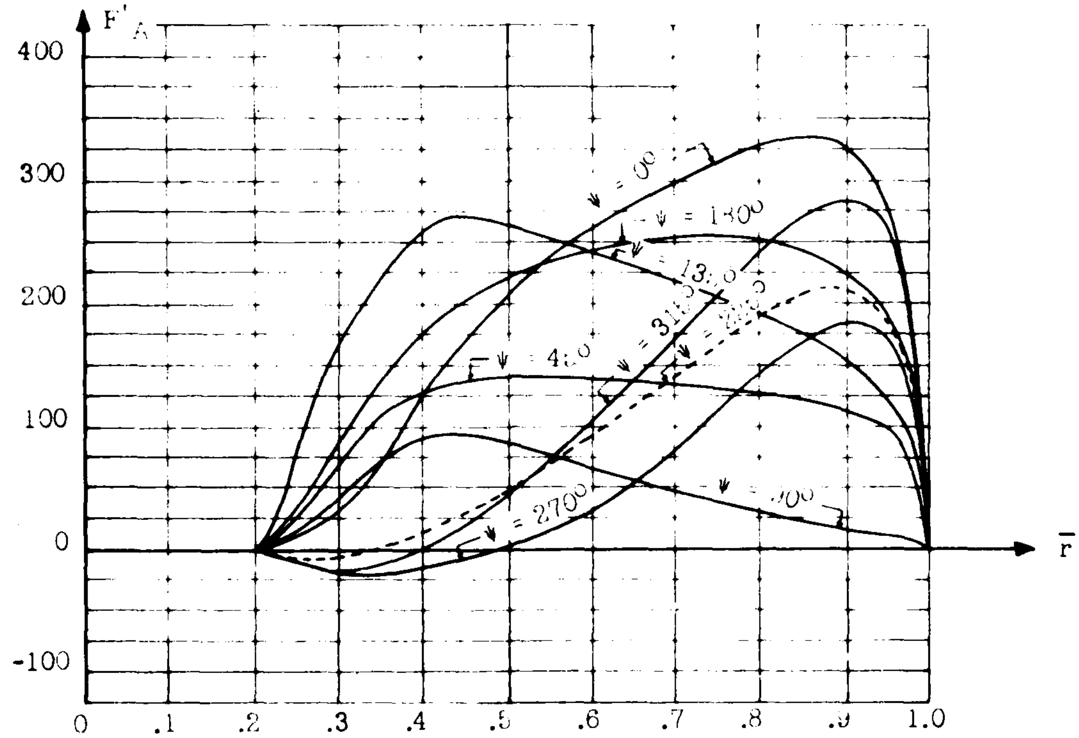

Figure 7.- N.C.2001 blade - distribution of aerodynamic forces F'_A on the rigid blade.

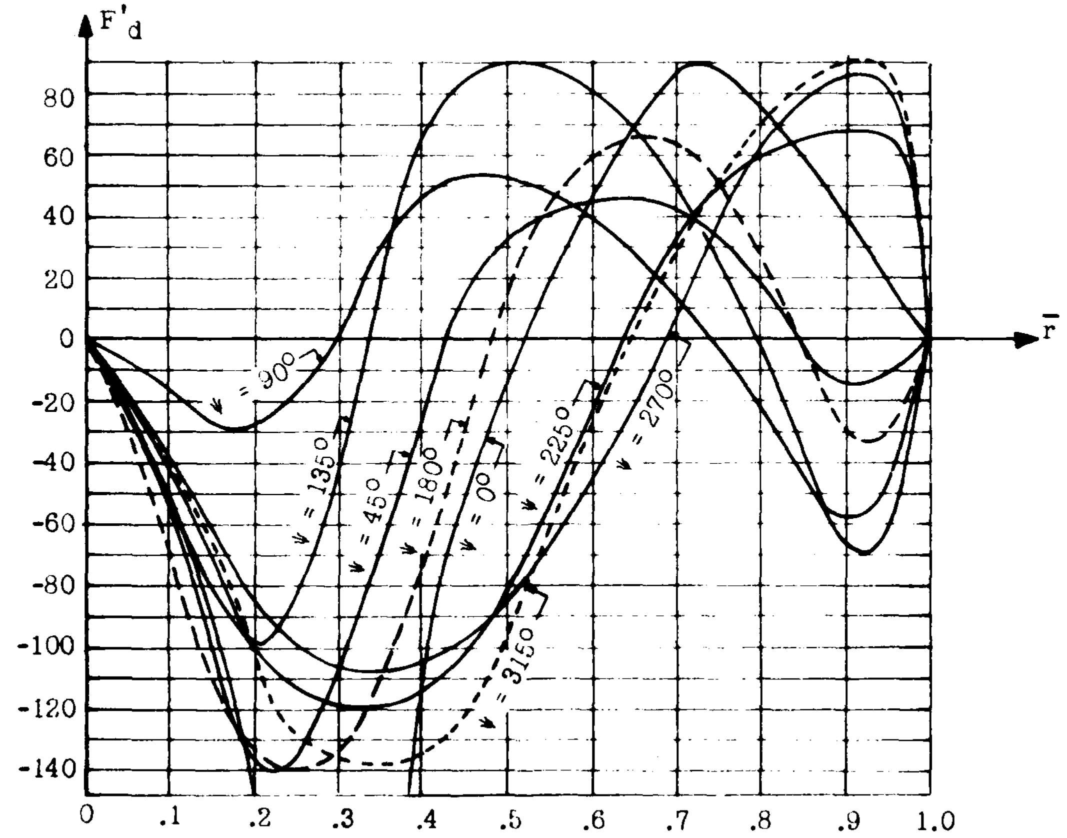

Figure 8.- N.C.2001 blade - total forces F'_d distributed over the rigid blade.

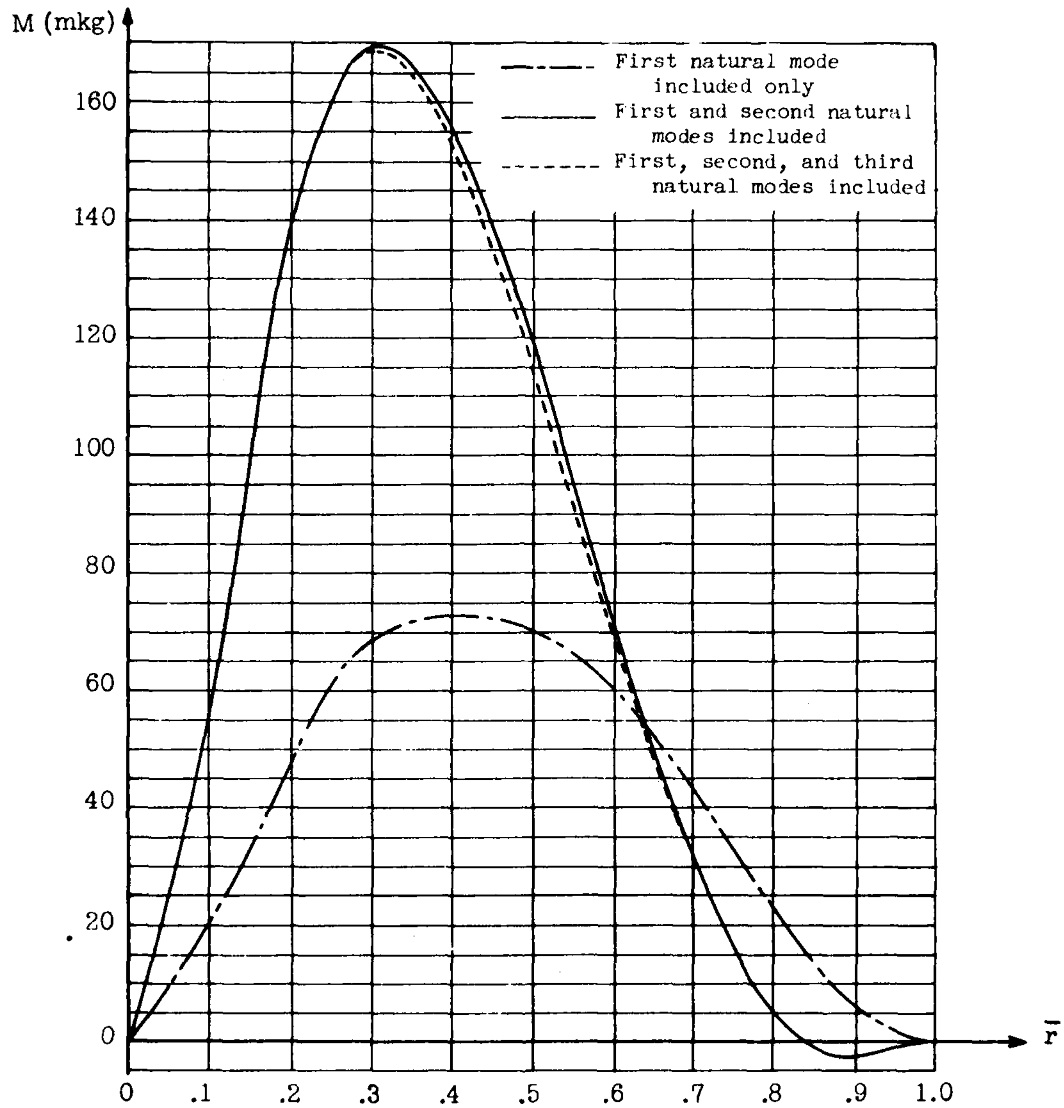

Figure 9.- N.C.2001 blade - blade bending moments for ψ = 0.

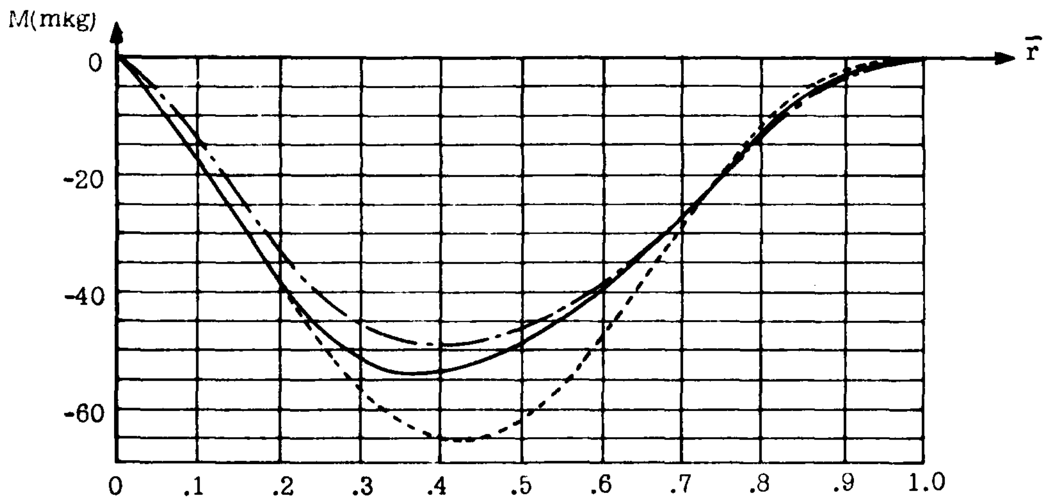

Figure 10.- N.C.2001 blade - blade bending moments for ψ = 90°.

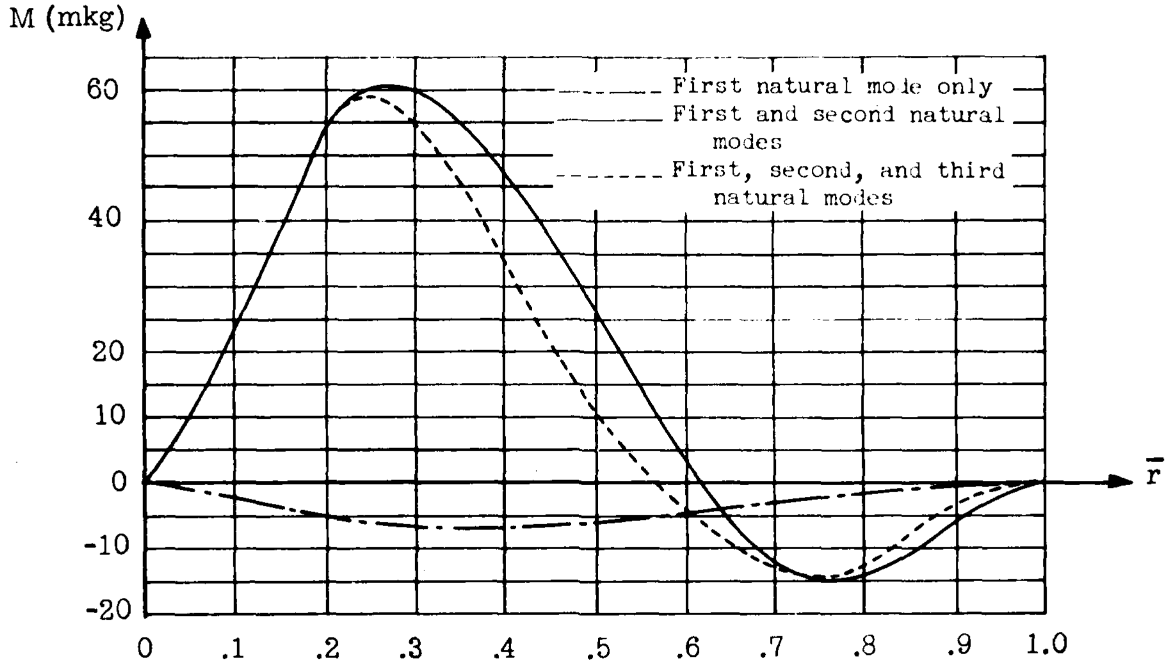

Figure 11.- N.C.2001 blade - blade bending moments for $\psi = 180^{\circ}$.

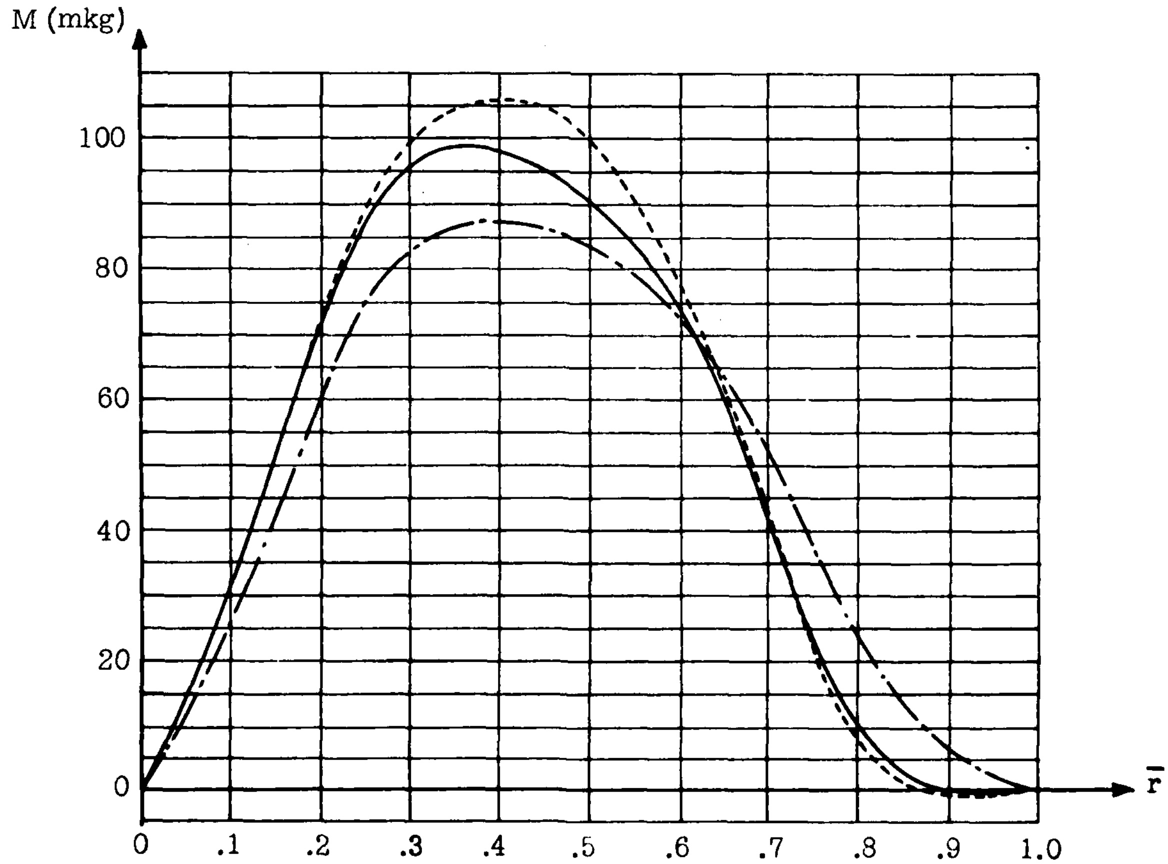

Figure 12.- N.C.2001 blade - blade bending moments for $\psi = 270^{\circ}$.

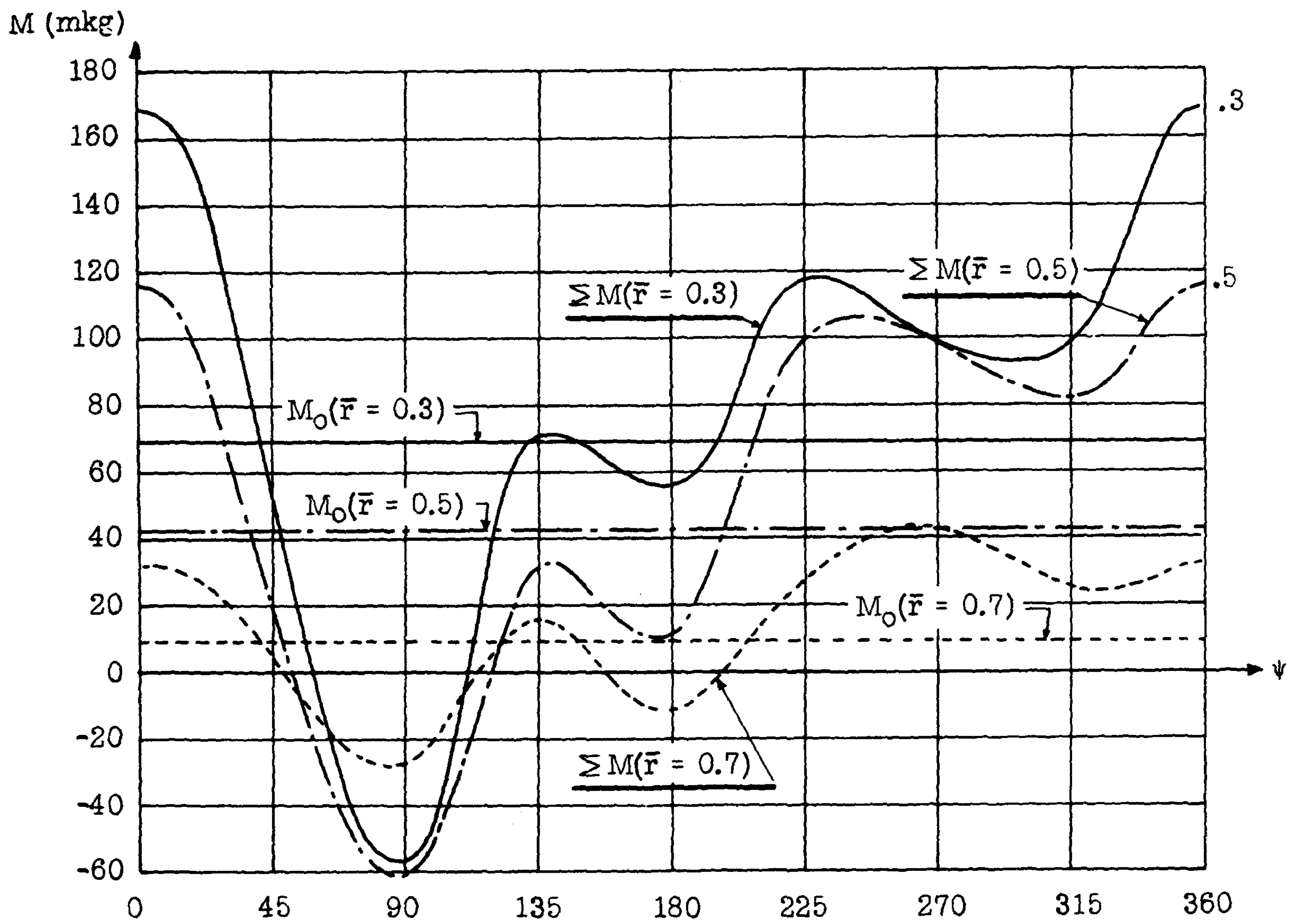

Figure 13.- N.C.2001 blade - blade bending moments plotted against ψ.

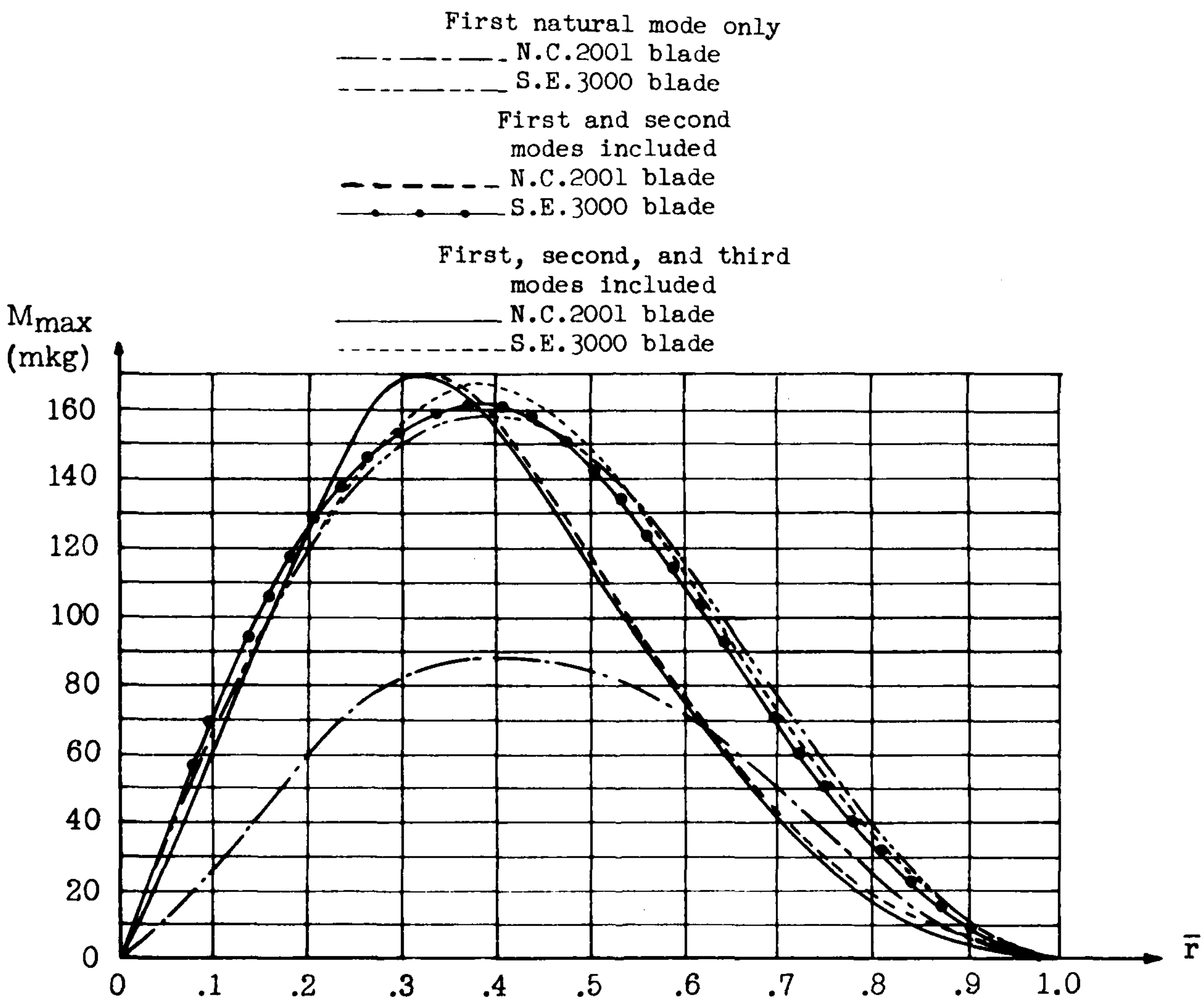

Figure 14.- Maximum bending moments.

de Guillenchmidt, P.

Calculation of the Bending Stresses in Helicopter Rotor Blades.

By P. de Guillenchmidt

NACA TM 1312
March 1951

(Abstract on Reverse Side)

Abstract

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